Kuwait University	**MATH 102**	June 14, 2009
Math. Dept.	**Final Exam**	Duration: two hours

Calculators and mobile telephones are not allowed.
Answer the following questions.

1. (4 pts) Let $f(x) = e^{2x} + e^{x}, \quad x \geq 0$.
(a) Show that f is one-to-one.
(b) Find the inverse function f^{-1}.
(c) Find the domain of f^{-1}.

2. (2 pts) Evaluate the limit $\displaystyle\lim_{x \to \frac{\pi}{2}^-} \frac{\sin^{-1}(\cos x)}{\ln(\sin x)}$

3. (4+4+4 pts) Evaluate the following integrals

$$(a) \int \frac{\sin(3x)}{2 + 3\cos^2(3x)} dx \qquad (b) \int \frac{\sqrt{x}}{\sqrt{1+\sqrt{x}}}\, dx \qquad (c) \int e^{-\frac{x}{2}} \ln(e^{x} + 1)\, dx$$

4. (3 pts) Determine whether the following improper integral is convergent or divergent, if convergent, find its value

$$\int_2^{\infty} \frac{x}{(x^2 - 4)^2}\, dx$$

5. (4 pts) Find the centroid of the region bounded by the curves $y = x^2$ and $x = y^2$.

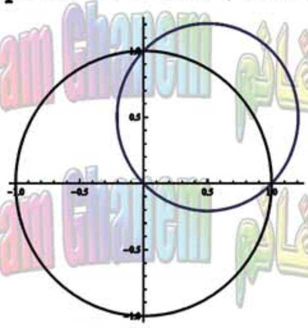
6. (4 pts) Consider the circle $r = \sin\theta$.
(a) Find the intersection points of the circle and the line $\theta = \dfrac{\pi}{4}$.
(b) Find the points on the circle, where the tangent line is parallel to the line $y = x$.

7. (3 pts) Find the length of the polar curve $r = \theta^2, \quad 0 \leq \theta \leq \pi$.

8. (3 pts) Find the surface area generated by rotating the parametric curve
$x = \cos t + \sin t, \quad y = \sin t - \cos t, \quad 0 \leq t \leq \pi/2$ around the y-axis.

9. (2 pts) Show that the polar equation $r = 2\sin\theta + 4\cos\theta$ represents a circle. Find its center and radius.

10. (3 pts) The graphs of the polar equations $r = \sin\theta + \cos\theta$ and $r = 1$ are shown below. Find the area of the region inside both circles.



1. (a) $f'(x) = 2e^{2x} + e^x > 0 \Rightarrow f$ is increasing $\Rightarrow f$ is one-to-one.

   (b) $y = e^{2x} + e^x \Rightarrow (e^x)^2 + e^x - y = 0 \Rightarrow e^x = \frac{-1+\sqrt{1+4y}}{2} \Rightarrow f^{-1}(x) = \ln(\frac{-1+\sqrt{1+4x}}{2})$.

   (c) $D_{f^{-1}} = R_f = [f(0)\,, \lim_{x\to\infty} f(x)) = [2,\infty)$.

2. The limit is of the form $\frac{0}{0}$, so by L'Hospital's Rule

$$\lim_{x\to(\frac{\pi}{2})^-} \frac{\sin^{-1}(\cos x)}{\ln(\sin x)} \overset{L'H}{=} \lim_{x\to(\frac{\pi}{2})^-} \frac{\frac{-\sin x}{\sqrt{1-\cos^2 x}}}{\frac{\cos x}{\sin x}} = \lim_{x\to(\frac{\pi}{2})^-} \frac{-\sin x}{\cos x} = -\infty.$$

3. (a) Let $u = \cos 3x \Rightarrow du = -3\sin 3x\,dx$, then

$$\int ... = \frac{-1}{9}\int \frac{du}{\frac{2}{3}+u^2} = \frac{-\sqrt{3}}{9\sqrt{2}}\tan^{-1}\frac{\sqrt{3}u}{\sqrt{2}} + c = \frac{-\sqrt{3}}{9\sqrt{2}}\tan^{-1}\frac{\sqrt{3}\cos x}{\sqrt{2}} + c.$$

   (b) Let $u = \sqrt{1+\sqrt{x}} \Rightarrow x = (u^2-1)^2 \Rightarrow dx = 4u(u^2-1)du$, then

$$\int ... = 4\int (u^2-1)^2 du = 4(\frac{u^5}{5} - 2\frac{u^3}{3} + u) + c = 4[\frac{(1+\sqrt{x})^{\frac{5}{2}}}{5} - 2\frac{(1+\sqrt{x})^{\frac{3}{2}}}{3} + (1+\sqrt{x})^{\frac{1}{2}}] + c.$$

   (c) By parts: take $u = \ln(e^x+1)$, $dv = e^{-\frac{x}{2}}$, then

$$\int ... = -2e^{-\frac{x}{2}}\ln(e^x+1) + 2\int \frac{e^{\frac{x}{2}}}{e^x+1}dx = -2\ e^{-\frac{x}{2}}\ln(e^x+1) + 4\ \tan^{-1}(e^{\frac{x}{2}}) + c.$$

4. $$\int_2^\infty ... = \int_2^3 ... + \int_3^\infty ... = \lim_{t\to 2^+}\int_t^3 \frac{x\,dx}{(x^2-4)^2} + \lim_{t\to\infty}\int_3^t \frac{x\,dx}{(x^2-4)^2}.$$

   The first term is $$\lim_{t\to 2^+}\int_t^3 \frac{x\,dx}{(x^2-4)^2} = \frac{-1}{2}\lim_{t\to 2^+}\left[\frac{1}{x^2-4}\right]_t^3 = \frac{-1}{2}\lim_{t\to 2^+}\left(\frac{1}{5} - \frac{1}{t^2-4}\right) = \infty,$$

   so the improper integral is divergent.

5. The region is symmetric about the line $y = x$, so the centroid is on the line $y = x$ and $\bar{x} = \bar{y}$.
   Now $\bar{x} = \frac{1}{A}\int_0^1 x(\sqrt{x} - x^2)dx = 3\int_0^1 (x^{3/2} - x^3)dx = \frac{9}{20}$, where $A = \int_0^1 (\sqrt{x} - x^2)dx = \frac{1}{3}$
   and the centroid is $(\bar{x},\bar{y}) = (\frac{9}{20}, \frac{9}{20})$.

6. (a) The two intersection points are $(\frac{\sqrt{2}}{2}, \frac{\pi}{4})$ and the pole.

   (b) For $r = \sin\theta$ the parametric equations are $x - \frac{1}{2}\sin 2\theta$, $y = \sin^2\theta$. Hence

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\sin 2\theta}{\cos 2\theta} = \tan 2\theta = 1$$

   and $\theta = \frac{\pi}{8}$ or $\frac{5\pi}{8}$, that is the points are $(\sin\frac{\pi}{8}, \frac{\pi}{8})$ and $(\sin\frac{5\pi}{8}, \frac{5\pi}{8})$.

7. For $r = \theta^2$, $0 \le \theta \le \pi$ the arc length is $\int_0^\pi \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\,d\theta = \int_0^\pi \sqrt{\theta^4 + 4\theta^2}\,d\theta$

$$= \int_0^\pi \theta\sqrt{\theta^2+4}\,d\theta = \frac{1}{2}\int_4^{4+\pi^2}\sqrt{u}\,du = \frac{1}{3}\left((4+\pi^2)^{3/2} - 4^{3/2}\right)$$

8. Let $x = \cos t + \sin t$, $y = \sin t - \cos t$, $0 \le t \le \pi/2$. Then

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt = \sqrt{2}\,dt \quad \text{and} \quad S = 2\pi\int x\,ds = 2\pi\int_0^{\pi/2}(\cos t + \sin t)\sqrt{2}\,dt = 4\sqrt{2}\,\pi$$

9. $r = 2\sin\theta + 4\cos\theta \Rightarrow r^2 = 2r\sin\theta + 4r\sin\theta$, then $x^2 + y^2 = 2y + 4x$ and completing the square $(x-2)^2 + (y-1)^2 = 5$, so the center of the circle is (2,1), the radius is $\sqrt{5}$.

10. The region consists of a quarter of the big circle and two pieces under the small circle, so

$$A = \frac{\pi}{4} + 2 \times \frac{1}{2} \int_{-\pi/4}^{0} (\sin\theta + \cos\theta)^2 \, d\theta = \frac{\pi}{4} + \int_{-\pi/4}^{0} (1 + \sin 2\theta) \, d\theta = \frac{\pi}{2} - \frac{1}{2}$$

Another solution: The radius of the small circle is $\frac{\sqrt{2}}{2}$, its area is $\frac{\pi}{2}$, then

$$A = \frac{\pi}{2} - \frac{1}{2} \int_{0}^{\frac{\pi}{2}} \left[(\sin\theta + \cos\theta)^2 - 1 \right] d\theta = \frac{\pi}{2} - \frac{1}{2}$$